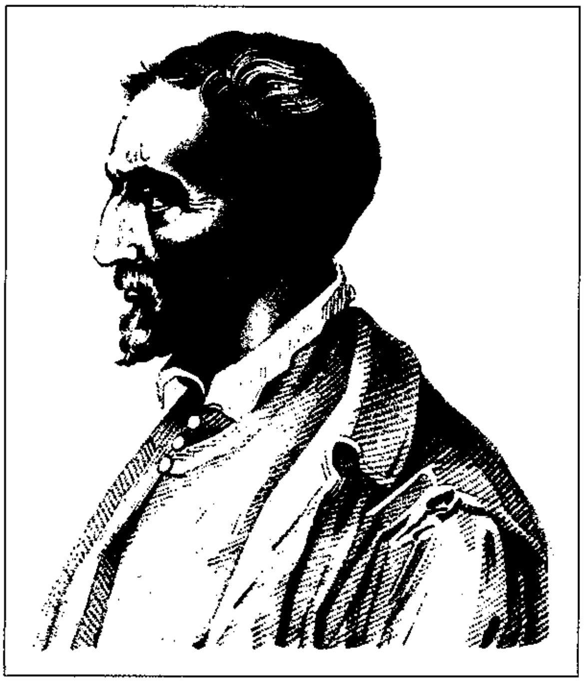
5 ALGEBRA 2



Girolamo Cardan (1501–1576) was a colourful character who became Professor of Mathematics at Milan. As well as being a distinguished academic, he was an astrologer, a physician, a gambler and a heretic, yet he received a pension from the Pope. His mathematical genius enabled him to open up the general theory of cubic and quartic equations, although a method for solving cubic equations which he claimed as his own was pirated from Niccolo Tartaglia.

10 Express direct and inverse variation in algebraic terms
20 Transform more complicated formulae
21 Manipulate algebraic fractions
23 Use indices, including fractional indices
24 Solve simple linear inequalities
25 Represent inequalities graphically and solve simple linear programming problems

5.1 Algebraic fractions

Simplifying fractions

Example

Simplify: (a) $\frac{32}{56}$ (b) $\frac{3a}{5a^2}$ (c) $\frac{3y+y^2}{6y}$

(a) $\frac{32}{56} = \frac{\cancel{8} \times 4}{\cancel{8} \times 7} = \frac{4}{7}$ (b) $\frac{3a}{5a^2} = \frac{3 \times \cancel{a}}{5 \times a \times \cancel{a}} = \frac{3}{5a}$ (c) $\frac{y(3+y)}{6y} = \frac{3+y}{6}$

Exercise 1

Simplify as far as possible, where you can.

1. $\frac{25}{35}$
2. $\frac{84}{96}$
3. $\frac{5y^2}{y}$
4. $\frac{y}{2y}$
5. $\frac{8x^2}{2x^2}$
6. $\frac{2x}{4y}$
7. $\frac{6y}{3y}$
8. $\frac{5ab}{10b}$
9. $\frac{8ab^2}{12ab}$
10. $\frac{7a^2b}{35ab^2}$
11. $\frac{(2a)^2}{4a}$
12. $\frac{7yx}{8xy}$
13. $\frac{5x+2x^2}{3x}$
14. $\frac{9x+3}{3x}$
15. $\frac{25+7}{25}$
16. $\frac{4a+5a^2}{5a}$
17. $\frac{3x}{4x-x^2}$
18. $\frac{5ab}{15a+10a^2}$
19. $\frac{5x+4}{8x}$
20. $\frac{12x+6}{6y}$
21. $\frac{5x+10y}{15xy}$
22. $\frac{18a-3ab}{6a^2}$
23. $\frac{4ab+8a^2}{2ab}$
24. $\frac{(2x)^2-8x}{4x}$

Example

Simplify:

(a) $\frac{x^2+x-6}{x^2+2x-3}=\frac{(x-2)\cancel{(x+3)}}{\cancel{(x+3)}(x-1)}=\frac{x-2}{x-1}$

(b) $\frac{x^2+3x-10}{x^2-4}=\frac{\cancel{(x-2)}(x+5)}{\cancel{(x-2)}(x+2)}=\frac{x+5}{x+2}$

(c) $\frac{3x^2-9x}{x^2-4x+3}=\frac{3x\cancel{(x-3)}}{(x-1)\cancel{(x-3)}}=\frac{3x}{x-1}$

Exercise 2

Simplify as far as possible:

1. $\frac{x^2+2x}{x^2-3x}$
2. $\frac{x^2-3x}{x^2-2x-3}$
3. $\frac{x^2+4x}{2x^2-10x}$
4. $\frac{x^2+6x+5}{x^2-x-2}$
5. $\frac{x^2-4x-21}{x^2-5x-14}$
6. $\frac{x^2+7x+10}{x^2-4}$
7. $\frac{x^2+x-2}{x^2-x}$
8. $\frac{3x^2-6x}{x^2+3x-10}$
9. $\frac{6x^2-2x}{12x^2-4x}$
10. $\frac{3x^2+15x}{x^2-25}$
11. $\frac{12x^2-20x}{4x^2}$
12. $\frac{x^2+x-6}{x^2+2x-3}$

Addition and subtraction of algebraic fractions

Example

Write as a single fraction:

(a) $\frac{2}{3}+\frac{3}{4}$ (b) $\frac{2}{x}+\frac{3}{y}$

Compare these two workings line for line:

(a) $\frac{2}{3}+\frac{3}{4}$; the L.C.M. of 3 and 4 is 12.

$$\therefore \quad \frac{2}{3}+\frac{3}{4}=\frac{8}{12}+\frac{9}{12}$$
$$=\frac{17}{12}$$

(b) $\frac{2}{x}+\frac{3}{y}$; the L.C.M. of x and y is xy.

$$\therefore \quad \frac{2}{x}+\frac{3}{y}=\frac{2y}{xy}+\frac{3x}{xy}$$
$$=\frac{2y+3x}{xy}$$

Exercise 3

Simplify the following:

1. $\frac{2}{5}+\frac{1}{5}$ **2.** $\frac{2x}{5}+\frac{x}{5}$ **3.** $\frac{2}{x}+\frac{1}{x}$

4. $\frac{1}{7}+\frac{3}{7}$ **5.** $\frac{x}{7}+\frac{3x}{7}$ **6.** $\frac{1}{7x}+\frac{3}{7x}$

7. $\frac{5}{8}+\frac{1}{4}$ **8.** $\frac{5x}{8}+\frac{x}{4}$ **9.** $\frac{5}{8x}+\frac{1}{4x}$

10. $\frac{2}{3}+\frac{1}{6}$ **11.** $\frac{2x}{3}+\frac{x}{6}$ **12.** $\frac{2}{3x}+\frac{1}{6x}$

13. $\frac{3}{4}+\frac{2}{5}$ **14.** $\frac{3x}{4}+\frac{2x}{5}$ **15.** $\frac{3}{4x}+\frac{2}{5x}$

16. $\frac{3}{4}-\frac{2}{3}$ **17.** $\frac{3x}{4}-\frac{2x}{3}$ **18.** $\frac{3}{4x}-\frac{2}{3x}$

19. $\frac{x}{2}+\frac{x+1}{3}$ **20.** $\frac{x-1}{3}+\frac{x+2}{4}$ **21.** $\frac{2x-1}{5}+\frac{x+3}{2}$

22. $\frac{x+1}{3}-\frac{2x+1}{4}$ **23.** $\frac{x-3}{3}-\frac{x-2}{5}$ **24.** $\frac{2x+1}{7}-\frac{x+2}{2}$

25. $\frac{1}{x}+\frac{2}{x+1}$ **26.** $\frac{3}{x-2}+\frac{4}{x}$ **27.** $\frac{5}{x-2}+\frac{3}{x+3}$

28. $\frac{7}{x+1}-\frac{3}{x+2}$ **29.** $\frac{2}{x+3}-\frac{5}{x-1}$ **30.** $\frac{3}{x-2}-\frac{4}{x+1}$

5.2 Changing the subject of a formula

The operations involved in solving ordinary linear equations are exactly the same as the operations required in changing the subject of a formula.

Example 1

(a) Solve the equation $3x + 1 = 12$.
(b) Make x the subject of the formula $Mx + B = A$.

(a) $3x + 1 = 12$
$$3x = 12 - 1$$
$$x = \frac{12 - 1}{3} = \frac{11}{3}$$

(b) $Mx + B = A$
$$Mx = A - B$$
$$x = \frac{A - B}{M}$$

Example 2

(a) Solve the equation $3(y - 2) = 5$.
(b) Make y the subject of the formula $x(y - a) = e$.

(a) $3(y - 2) = 5$
$$3y - 6 = 5$$
$$3y = 11$$
$$y = \frac{11}{3}$$

(b) $x(y - a) = e$
$$xy - xa = e$$
$$xy = e + xa$$
$$y = \frac{e + xa}{x}$$

Exercise 4

Make x the subject of the following:

1. $2x = 5$	**2.** $7x = 21$	**3.** $Ax = B$
4. $Nx = T$	**5.** $Mx = K$	**6.** $xy = 4$
7. $Bx = C$	**8.** $4x = D$	**9.** $9x = \mathrm{T} + \mathrm{N}$
10. $Ax = B - R$	**11.** $Cx = R + T$	**12.** $Lx = N - R^2$
13. $R - S^2 = Nx$	**14.** $x + 5 = 7$	**15.** $x + 10 = 3$
16. $x + A = T$	**17.** $x + B = S$	**18.** $N = x + D$
19. $M = x + B$	**20.** $L = x + D^2$	**21.** $N^2 + x = T$
22. $L + x = N + M$	**23.** $Z + x = R - S$	**24.** $x - 5 = 2$
25. $x - R = A$	**26.** $x - A = E$	**27.** $F = x - B$
28. $F^2 = x - B^2$	**29.** $x - D = A + B$	**30.** $x - E = A^2$

Make y the subject of the following:

31. $L = y - B$
32. $N = y - T$
33. $3y + 1 = 7$
34. $2y - 4 = 5$
35. $Ay + C = N$
36. $By + D = L$
37. $Dy + E = F$
38. $Ny - F = H$
39. $Yy - Z = T$
40. $Ry - L = B$
41. $Vy + m = Q$
42. $ty - m = n + a$
43. $qy + n = s - t$
44. $ny - s^2 = t$
45. $V^2y + b = c$
46. $r = ny - 6$
47. $s = my + d$
48. $t = my - b$
49. $j = my + c$
50. $2(y + 1) = 6$
51. $3(y - 1) = 5$
52. $A(y + B) = C$
53. $D(y + E) = F$
54. $h(y + n) = a$
55. $b(y - d) = q$
56. $n = r(y + t)$
57. $t(y - 4) = b$
58. $z = S(y + t)$
59. $s = v(y - d)$
60. $g = m(y + n)$

Example 1

(a) Solve the equation $\dfrac{3a + 1}{2} = 4$.

(b) Make a the subject of the formula $\dfrac{na + b}{m} = n$.

(a)
$$\frac{3a + 1}{2} = 4$$
$$3a + 1 = 8$$
$$3a = 7$$
$$a = \frac{7}{3}$$

(b)
$$\frac{na + b}{m} = n$$
$$na + b = mn$$
$$na = mn - b$$
$$a = \frac{mn - b}{n}$$

Example 2

Make a the subject of the formula

$$x - na = y$$

Make the 'a' term positive

$$x = y + na$$
$$x - y = na$$
$$\frac{x - y}{n} = a$$

Exercise 5

Make a the subject.

1. $\dfrac{a}{4} = 3$
2. $\dfrac{a}{5} = 2$
3. $\dfrac{a}{D} = B$
4. $\dfrac{a}{B} = T$
5. $\dfrac{a}{N} = R$
6. $b = \dfrac{a}{m}$
7. $\dfrac{a - 2}{4} = 6$
8. $\dfrac{a - A}{B} = T$
9. $\dfrac{a - D}{N} = A$
10. $\dfrac{a + Q}{N} = B^2$
11. $g = \dfrac{a - r}{e}$
12. $\dfrac{2a + 1}{5} = 2$
13. $\dfrac{Aa + B}{C} = D$
14. $\dfrac{na + m}{p} = q$
15. $\dfrac{ra - t}{S} = v$

16. $\dfrac{za-m}{q}=t$
17. $\dfrac{m+Aa}{b}=c$
18. $A=\dfrac{Ba+D}{E}$
19. $n=\dfrac{ea-f}{h}$
20. $q=\dfrac{ga+b}{r}$
21. $6-a=2$
22. $7-a=9$
23. $5=7-a$
24. $A-a=B$
25. $C-a=E$
26. $D-a=H$
27. $n-a=m$
28. $t=q-a$
29. $b=s-a$
30. $v=r-a$
31. $t=m-a$
32. $5-2a=1$
33. $T-Xa=B$
34. $M-Na=Q$
35. $V-Ma=T$
36. $L=N-Ra$
37. $r=v^2-ra$
38. $t^2=w-na$
39. $n-qa=2$
40. $\dfrac{3-4a}{2}=1$
41. $\dfrac{5-7a}{3}=2$
42. $\dfrac{B-Aa}{D}=E$
43. $\dfrac{D-Ea}{N}=B$
44. $\dfrac{h-fa}{b}=x$
45. $\dfrac{v^2-ha}{C}=d$
46. $\dfrac{M(a+B)}{N}=T$
47. $\dfrac{f(Na-e)}{m}=B$
48. $\dfrac{T(M-a)}{E}=F$
49. $\dfrac{y(x-a)}{z}=t$
50. $\dfrac{k^2(m-a)}{x}=x$

Example 1

(a) Solve the equation $\dfrac{4}{z}=7$.

(b) Make z the subject of the formula $\dfrac{n}{z}=k$.

(a) $\dfrac{4}{z}=7$

$4=7z$

$\dfrac{4}{7}=z$

(b) $\dfrac{n}{z}=k$

$n=kz$

$\dfrac{n}{k}=z$

Example 2

Make t the subject of the formula $\dfrac{x}{t}+m=a$.

$$\frac{x}{t}=a-m$$

$$x=(a-m)t$$

$$\frac{x}{(a-m)}=t$$

Exercise 6

Make a the subject.

1. $\dfrac{7}{a}=14$
2. $\dfrac{5}{a}=3$
3. $\dfrac{B}{a}=C$
4. $\dfrac{T}{a}=X$
5. $\dfrac{M}{a}=B$
6. $m=\dfrac{n}{a}$
7. $t=\dfrac{v}{a}$
8. $\dfrac{n}{a}=\sin 20°$

9. $\frac{7}{a} = \cos 30°$ **10.** $\frac{B}{a} = x$ **11.** $\frac{5}{a} = \frac{3}{4}$ **12.** $\frac{N}{a} = \frac{B}{D}$

13. $\frac{H}{a} = \frac{N}{M}$ **14.** $\frac{t}{a} = \frac{b}{e}$ **15.** $\frac{v}{a} = \frac{m}{s}$ **16.** $\frac{t}{b} = \frac{m}{a}$

17. $\frac{5}{a+1} = 2$ **18.** $\frac{7}{a-1} = 3$ **19.** $\frac{B}{a+D} = C$ **20.** $\frac{Q}{a-C} = T$

21. $\frac{V}{a-T} = D$ **22.** $\frac{L}{Ma} = B$ **23.** $\frac{N}{Ba} = C$ **24.** $\frac{m}{ca} = d$

25. $t = \frac{b}{c-a}$ **26.** $x = \frac{z}{y-a}$

Make x the subject.

27. $\frac{2}{x} + 1 = 3$ **28.** $\frac{5}{x} - 2 = 4$ **29.** $\frac{A}{x} + B = C$ **30.** $\frac{V}{x} + G = H$

31. $\frac{r}{x} - t = n$ **32.** $q = \frac{b}{x} + d$ **33.** $t = \frac{m}{x} - n$ **34.** $h = d - \frac{b}{x}$

35. $C - \frac{d}{x} = e$ **36.** $r - \frac{m}{x} = e^2$ **37.** $t^2 = b - \frac{n}{x}$ **38.** $\frac{d}{x} + b = mn$

39. $\frac{M}{x+q} - N = 0$ **40.** $\frac{Y}{x-c} - T = 0$ **41.** $3M = M + \frac{N}{P+x}$ **42.** $A = \frac{B}{c+x} - 5A$

43. $\frac{K}{Mx} + B = C$ **44.** $\frac{z}{xy} - z = y$ **45.** $\frac{m^2}{x} - n = -p$ **46.** $t = w - \frac{q}{x}$

Example

Make x the subject of the formulae.

(a) $\sqrt{(x^2 + A)} = B$
$x^2 + A = B^2$ (square both sides)
$x^2 = B^2 - A$
$x = \pm\sqrt{(B^2 - A)}$

(b) $(Ax - B)^2 = M$
$Ax - B = \pm\sqrt{M}$ (square root both sides)
$Ax = B \pm \sqrt{M}$
$x = \frac{B \pm \sqrt{M}}{A}$

(c) $\sqrt{(R - x)} = T$
$R - x = T^2$
$R = T^2 + x$
$R - T^2 = x$

Exercise 7

Make x the subject.

1. $\sqrt{x} = 2$ **2.** $\sqrt{(x+1)} = 5$ **3.** $\sqrt{(x-2)} = 3$ **4.** $\sqrt{(x+a)} = B$

5. $\sqrt{(x+C)} = D$ **6.** $\sqrt{(x-E)} = H$ **7.** $\sqrt{(ax+b)} = c$ **8.** $\sqrt{(x-m)} = a$

9. $b = \sqrt{(gx-t)}$ **10.** $r = \sqrt{(b-x)}$ **11.** $\sqrt{(d-x)} = t$ **12.** $b = \sqrt{(x-d)}$

13. $c = \sqrt{(n-x)}$ **14.** $f = \sqrt{(b-x)}$ **15.** $g = \sqrt{(c-x)}$ **16.** $\sqrt{(M-Nx)} = P$

17. $\sqrt{(Ax+B)} = \sqrt{D}$ **18.** $\sqrt{(x-D)} = A^2$ **19.** $x^2 = g$ **20.** $x^2 + 1 = 17$

21. $x^2 = B$
22. $x^2 + A = B$
23. $x^2 - A = M$
24. $b = a + x^2$
25. $C - x^2 = m$
26. $n = d - x^2$
27. $mx^2 = n$
28. $b = ax^2$

Make k the subject.

29. $\frac{kz}{a} = t$
30. $ak^2 - t = m$
31. $n = a - k^2$
32. $\sqrt{(k^2 - 4)} = 6$
33. $\sqrt{(k^2 - A)} = B$
34. $\sqrt{(k^2 + y)} = x$
35. $t = \sqrt{(m + k^2)}$
36. $2\sqrt{(k + 1)} = 6$
37. $A\sqrt{(k + B)} = M$
38. $\sqrt{\left(\frac{M}{k}\right)} = N$
39. $\sqrt{\left(\frac{N}{k}\right)} = B$
40. $\sqrt{(a - k)} = b$
41. $\sqrt{(a^2 - k^2)} = t$
42. $\sqrt{(m - k^2)} = x$
43. $2\pi\sqrt{(k + t)} = 4$
44. $A\sqrt{(k + 1)} = B$
45. $\sqrt{(ak^2 - b)} = C$
46. $a\sqrt{(k^2 - x)} = b$
47. $k^2 + b = x^2$
48. $\frac{k^2}{a} + b = c$
49. $\sqrt{(c^2 - ak)} = b$
50. $\frac{m}{k^2} = a + b$

Example

Make x the subject of the formulae.

(a)
$$\begin{aligned} Ax - B &= Cx + D \\ Ax - Cx &= D + B \\ x(A - C) &= D + B \text{ (factorise)} \\ x &= \frac{D + B}{A - C} \end{aligned}$$

(b)
$$\begin{aligned} x + a &= \frac{x + b}{c} \\ c(x + a) &= x + b \\ cx + ca &= x + b \\ cx - x &= b - ca \\ x(c - 1) &= b - ca \text{ (factorise)} \\ x &= \frac{b - ca}{c - 1} \end{aligned}$$

Exercise 8

Make y the subject.

1. $5(y - 1) = 2(y + 3)$
2. $7(y - 3) = 4(3 - y)$
3. $Ny + B = D - Ny$
4. $My - D = E - 2My$
5. $ay + b = 3b + by$
6. $my - c = e - ny$
7. $xy + 4 = 7 - ky$
8. $Ry + D = Ty + C$
9. $ay - x = z + by$
10. $m(y + a) = n(y + b)$
11. $x(y - b) = y + d$
12. $\frac{a - y}{a + y} = b$
13. $\frac{1 - y}{1 + y} = \frac{c}{d}$
14. $\frac{M - y}{M + y} = \frac{a}{b}$
15. $m(y + n) = n(n - y)$
16. $y + m = \frac{2y - 5}{m}$
17. $y - n = \frac{y + 2}{n}$
18. $y + b = \frac{ay + e}{b}$
19. $\frac{ay + x}{x} = 4 - y$
20. $c - dy = e - ay$
21. $y(a - c) = by + d$
22. $y(m + n) = a(y + b)$
23. $t - ay = s - by$
24. $\frac{y + x}{y - x} = 3$

25. $\dfrac{v-y}{v+y}=\dfrac{1}{2}$

26. $y(b-a)=a(y+b+c)$

27. $\sqrt{\left(\dfrac{y+x}{y-x}\right)}=2$

28. $\sqrt{\left(\dfrac{z+y}{z-y}\right)}=\dfrac{1}{3}$

29. $\sqrt{\left[\dfrac{m(y+n)}{y}\right]}=p$

30. $n-y=\dfrac{4y-n}{m}$

Example

Make w the subject of the formula $\sqrt{\left(\dfrac{w}{w+a}\right)}=c$.

Squaring both sides, $\dfrac{w}{w+a}=c^2$

Multiplying by $(w+a)$, $w=c^2(w+a)$

$$w=c^2w+c^2a$$
$$w-c^2w=c^2a$$
$$w(1-c^2)=c^2a$$
$$w=\frac{c^2a}{1-c^2}$$

Exercise 9

Make the letter in square brackets the subject.

1. $ax+by+c=0$ [x]
2. $\sqrt{\{a(y^2-b)\}}=e$ [y]
3. $\dfrac{\sqrt{(k-m)}}{n}=\dfrac{1}{m}$ [k]
4. $a-bz=z+b$ [z]
5. $\dfrac{x+y}{x-y}=2$ [x]
6. $\sqrt{\left(\dfrac{a}{z}-c\right)}=e$ [z]
7. $lm+mn+a=0$ [n]
8. $t=2\pi\sqrt{\left(\dfrac{d}{g}\right)}$ [d]
9. $t=2\pi\sqrt{\left(\dfrac{d}{g}\right)}$ [g]
10. $\sqrt{(x^2+a)}=2x$ [x]
11. $\sqrt{\left\{\dfrac{b(m^2+a)}{e}\right\}}=t$ [m]
12. $\sqrt{\left(\dfrac{x+1}{x}\right)}=a$ [x]
13. $a+b-mx=0$ [m]
14. $\sqrt{(a^2+b^2)}=x^2$ [a]
15. $\dfrac{a}{k}+b=\dfrac{c}{k}$ [k]
16. $a-y=\dfrac{b+y}{a}$ [y]
17. $G=4\pi\sqrt{(x^2+T^2)}$ [x]
18. $M(ax+by+c)=0$ [y]
19. $x=\sqrt{\left(\dfrac{y-1}{y+1}\right)}$ [y]
20. $a\sqrt{\left(\dfrac{x^2-n}{m}\right)}=\dfrac{a^2}{b}$ [x]
21. $\dfrac{M}{N}+E=\dfrac{P}{N}$ [N]
22. $\dfrac{Q}{P-x}=R$ [x]
23. $\sqrt{(z-ax)}=t$ [a]
24. $e+\sqrt{(x+f)}=g$ [x]
25. $\dfrac{m(ny-e^2)}{p}+n=5n$ [y]

5.3 Variation

Direct variation

There are several ways of expressing a relationship between two quantities x and y. Here are some examples.

x varies as y
x varies directly as y
x is proportional to y

These three all mean the same and they are written in symbols as follows.

$x \propto y$

The '$\propto$' sign can always be replaced by '$= k$' where k is a constant:

$x = ky$

Suppose $x = 3$ when $y = 12$;
then $3 = k \times 12$
and $k = \frac{1}{4}$
We can then write $x = \frac{1}{4}y$, and this allows us to find the value of x for any value of y and *vice versa*.

Example 1

y varies as z, and $y = 2$ when $z = 5$; find
(a) the value of y when $z = 6$
(b) the value of z when $y = 5$

Because $y \propto z$, then $y = kz$ where k is a constant.
$y = 2$ when $z = 5$
$2 = k \times 5$
$k = \frac{2}{5}$
So $y = \frac{2}{5}z$

(a) When $z = 6$, $y = \frac{2}{5} \times 6 = 2\frac{2}{5}$
(b) When $y = 5$, $5 = \frac{2}{5}z$
$z = \frac{25}{2} = 12\frac{1}{2}$

Example 2

The value V of a diamond is proportional to the square of its weight W. If a diamond weighing 10 grams is worth \$200, find:
(a) the value of a diamond weighing 30 grams
(b) the weight of a diamond worth \$5000.

$V \propto W^2$
or $V = kW^2$ where k is a constant.
$V = 200$ when $W = 10$
$\therefore$ $200 = k \times 10^2$
$k = 2$
So $V = 2W^2$

(a) When $W = 30$,
$$V = 2 \times 30^2 = 2 \times 900$$
$$V = £1800$$
So a diamond of weight 30 grams is worth \$1800.

(b) When $V = 5000$,
$$5000 = 2 \times W^2$$
$$W^2 = \frac{5000}{2} = 2500$$
$$W = \sqrt{2500} = 50$$
So a diamond of value \$5000 weighs 50 grams.

Exercise 10

1. Rewrite the statement connecting each pair of variables using a constant k instead of '$\propto$'.

(a) $S \propto e$ (b) $v \propto t$ (c) $x \propto z^2$
(d) $y \propto \sqrt{x}$ (e) $T \propto \sqrt{L}$ (f) $C \propto r$
(g) $A \propto r^2$ (h) $V \propto r^3$

2. y varies as t. If $y = 6$ when $t = 4$, calculate:
(a) the value of y, when $t = 6$
(b) the value of t, when $y = 4$.

3. z is proportional to m. If $z = 20$ when $m = 4$, calculate:
(a) the value of z, when $m = 7$
(b) the value of m, when $z = 55$.

4. A varies directly as r^2. If $A = 12$, when $r = 2$, calculate:
(a) the value of A, when $r = 5$
(b) the value of r, when $A = 48$.

5. Given that $z \propto x$, copy and complete the table.

x	1	3		$5\frac{1}{2}$
z	4		16	

6. Given that $V \propto r^3$, copy and complete the table.

r	1	2		$1\frac{1}{2}$
V	4		256	

7. Given that $w \propto \sqrt{h}$, copy and complete the table.

h	4	9		$2\frac{1}{4}$
w	6		15	

8. s is proportional to $(v - 1)^2$. If $s = 8$, when $v = 3$, calculate:
(a) the value of s, when $v = 4$
(b) the value of v, when $s = 2$.

9. m varies as $(d + 3)$. If $m = 28$ when $d = 1$, calculate:
   (a) the value of m, when $d = 3$
   (b) the value of d, when $m = 49$.

10. The pressure of the water P at any point below the surface of the sea varies as the depth of the point below the surface d. If the pressure is 200 newtons/cm^2 at a depth of 3 m, calculate the pressure at a depth of 5 m.

11. The distance d through which a stone falls from rest is proportional to the square of the time taken t. If the stone falls 45 m in 3 seconds, how far will it fall in 6 seconds?
    How long will it take to fall 20 m?

12. The energy E stored in an elastic band varies as the square of the extension x. When the elastic is extended by 3 cm, the energy stored is 243 joules. What is the energy stored when the extension is 5 cm? What is the extension when the stored energy is 36 joules?

13. In the first few days of its life, the length of an earthworm l is thought to be proportional to the square root of the number of hours n which have elapsed since its birth. If a worm is 2 cm long after 1 hour, how long will it be after 4 hours?
    How long will it take to grow to a length of 14 cm?

14. It is well known that the number of golden eggs which a goose lays in a week varies as the cube root of the average number of hours of sleep she has. When she has 8 hours sleep, she lays 4 golden eggs. How long does she sleep when she lays 5 golden eggs?

15. The resistance to motion of a car is proportional to the square of the speed of the car. If the resistance is 4000 newtons at a speed of 20 m/s, what is the resistance at a speed of 30 m/s?
    At what speed is the resistance 6250 newtons?

16. A road research organisation recently claimed that the damage to road surfaces was proportional to the fourth power of the axle load. The axle load of a 44 ton HGV is about 15 times that of a car. Calculate the ratio of the damage to road surfaces made by a 44-ton HGV and a car.

Inverse variation

There are several ways of expressing an inverse relationship between two variables,

x varies inversely as y
x is inversely proportional to y.

We write $x \propto \frac{1}{y}$ for both statements and proceed using the method outlined in the previous section.

Example

z is inversely proportional to t^2 and $z = 4$ when $t = 1$. Calculate:
(a) z when $t = 2$
(b) t when $z = 16$.

We have $z \propto \dfrac{1}{t^2}$

or $z = k \times \dfrac{1}{t^2}$ (k is a constant)

$z = 4$ when $t = 1$,

$\therefore \quad 4 = k\left(\dfrac{1}{1^2}\right)$

so $k = 4$

$\therefore \quad z = 4 \times \dfrac{1}{t^2}$

(a) when $t = 2$, $z = 4 \times \dfrac{1}{2^2} = 1$

(b) when $z = 16$, $16 = 4 \times \dfrac{1}{t^2}$

$$16t^2 = 4$$
$$t^2 = \tfrac{1}{4}$$
$$t = \pm\tfrac{1}{2}$$

Exercise 11

1. Rewrite the statements connecting the variables using a constant of variation, k.
   (a) $x \propto \dfrac{1}{y}$ (b) $s \propto \dfrac{1}{t^2}$ (c) $t \propto \dfrac{1}{\sqrt{q}}$
   (d) m varies inversely as w
   (e) z is inversely proportional to t^2.
2. b varies inversely as e. If $b = 6$ when $e = 2$, calculate:
   (a) the value of b when $e = 12$
   (b) the value of e when $b = 3$.
3. q varies inversely as r. If $q = 5$ when $r = 2$, calculate:
   (a) the value of q when $r = 4$
   (b) the value of r when $q = 20$.
4. x is inversely proportional to y^2. If $x = 4$ when $y = 3$, calculate:
   (a) the value of x when $y = 1$
   (b) the value of y when $x = 2\tfrac{1}{4}$.
5. R varies inversely as v^2. If $R = 120$ when $v = 1$, calculate:
   (a) the value of R when $v = 10$
   (b) the value of v when $R = 30$.
6. T is inversely proportional to x^2. If $T = 36$ when $x = 2$, calculate:
   (a) the value of T when $x = 3$
   (b) the value of x when $T = 1{\cdot}44$.

7. p is inversely proportional to $\sqrt{y}$. If $p = 1{\cdot}2$ when $y = 100$, calculate:
(a) the value of p when $y = 4$
(b) the value of y when $p = 3$.

8. y varies inversely as z. If $y = \frac{1}{8}$ when $z = 4$, calculate:
(a) the value of y when $z = 1$
(b) the value of z when $y = 10$.

9. Given that $z \propto \dfrac{1}{y}$, copy and complete the table:

y	2	4		$\frac{1}{4}$
z	8		16	

10. Given that $v \propto \dfrac{1}{t^2}$, copy and complete the table:

t	2	5		10
v	25		$\frac{1}{4}$	

11. Given that $r \propto \dfrac{1}{\sqrt{x}}$, copy and complete the table:

x	1	4		
r	12		$\frac{3}{4}$	2

12. e varies inversely as $(y - 2)$. If $e = 12$ when $y = 4$, find
(a) e when $y = 6$ (b) y when $e = \frac{1}{2}$.

13. M is inversely proportional to the square of l.
If $M = 9$ when $l = 2$, find:
(a) M when $l = 10$ (b) l when $M = 1$.

14. Given $z = \dfrac{k}{x^n}$, find k and n, then copy and complete the table.

x	1	2	4	
z	100	$12\frac{1}{2}$		$\frac{1}{10}$

15. Given $y = \dfrac{k}{\sqrt[n]{v}}$, find k and n, then copy and complete the table.

v	1	4	36	
y	12	6		$\frac{3}{25}$

16. The volume V of a given mass of gas varies inversely as the pressure P. When $V = 2\,\text{m}^3$, $P = 500\,\text{N/m}^2$. Find the volume when the pressure is $400\,\text{N/m}^2$. Find the pressure when the volume is $5\,\text{m}^3$.

17. The number of hours N required to dig a certain hole is inversely proportional to the number of men available x. When 6 men are digging, the hole takes 4 hours. Find the time taken when 8 men are available. If it takes $\frac{1}{2}$ hour to dig the hole, how many men are there?

18. The life expectancy L of a rat varies inversely as the square of the density d of poison distributed around his home. When the density of poison is $1\,\text{g/m}^2$ the life expectancy is 50 days. How long will he survive if the density of poison is:
(a) $5\,\text{g/m}^2$? (b) $\frac{1}{2}\,\text{g/m}^2$?

19. The force of attraction F between two magnets varies inversely as the square of the distance d between them. When the magnets are 2 cm apart, the force of attraction is 18 newtons. How far apart are they if the attractive force is 2 newtons?

5.4 Indices

Rules of indices

1. $a^n \times a^m = a^{n+m}$ e.g. $7^2 \times 7^4 = 7^6$
2. $a^n \div a^m = a^{n-m}$ e.g. $6^6 \div 6^2 = 6^4$
3. $(a^n)^m = a^{nm}$ e.g. $(3^2)^5 = 3^{10}$

Also, $a^{-n} = \dfrac{1}{a^n}$ e.g. $5^{-2} = \dfrac{1}{5^2}$

$a^{\frac{1}{n}}$ means 'the nth root of a' e.g. $9^{\frac{1}{2}} = \sqrt[2]{9}$

$a^{\frac{m}{n}}$ means 'the nth root of a raised to the power m'
e.g. $4^{\frac{3}{2}} = (\sqrt{4})^3 = 8$

Example

Simplify:
(a) $x^7 \times x^{13}$ (b) $x^3 \div x^7$
(c) $(x^4)^3$ (d) $(3x^2)^3$
(e) $(2x^{-1})^2 \div x^{-5}$ (f) $3y^2 \times 4y^3$

(a) $x^7 \times x^{13} = x^{7+13} = x^{20}$

(b) $x^3 \div x^7 = x^{3-7} = x^{-4} = \dfrac{1}{x^4}$

(c) $(x^4)^3 = x^{12}$

(d) $(3x^2)^3 = 3^3 \times (x^2)^3 = 27x^6$

(e) $(2x^{-1})^2 \div x^{-5} = 4x^{-2} \div x^{-5}$
$= 4x^{(-2--5)}$
$= 4x^3$

(f) $3y^2 \times 4y^3 = 12y^5$

Exercise 12

Express in index form:

1. $3 \times 3 \times 3 \times 3$ **2.** $4 \times 4 \times 5 \times 5 \times 5$ **3.** $3 \times 7 \times 7 \times 7$

4. $2 \times 2 \times 2 \times 7$ **5.** $\dfrac{1}{10 \times 10 \times 10}$ **6.** $\dfrac{1}{2 \times 2 \times 3 \times 3 \times 3}$

7. $\sqrt{15}$ **8.** $\sqrt[3]{3}$ **9.** $\sqrt[5]{10}$

10. $(\sqrt{5})^3$

Simplify:

11. $x^3 \times x^4$ **12.** $y^6 \times y^7$ **13.** $z^7 \div z^3$ **14.** $z^{50} \times z^{50}$
15. $m^3 \div m^2$ **16.** $e^{-3} \times e^{-2}$ **17.** $y^{-2} \times y^4$ **18.** $w^4 \div w^{-2}$
19. $y^{\frac{1}{2}} \times y^{\frac{1}{2}}$ **20.** $(x^2)^5$ **21.** $x^{-2} \div x^{-2}$ **22.** $w^{-3} \times w^{-2}$
23. $w^{-7} \times w^2$ **24.** $x^3 \div x^{-4}$ **25.** $(a^2)^4$ **26.** $(k^{\frac{1}{2}})^6$
27. $e^{-4} \times e^4$ **28.** $x^{-1} \times x^{30}$ **29.** $(y^4)^{\frac{1}{2}}$ **30.** $(x^{-3})^{-2}$
31. $z^2 \div z^{-2}$ **32.** $t^{-3} \div t$ **33.** $(2x^3)^2$ **34.** $(4y^5)^2$
35. $2x^2 \times 3x^2$ **36.** $5y^3 \times 2y^2$ **37.** $5a^3 \times 3a$ **38.** $(2a)^3$
39. $3x^3 \div x^3$ **40.** $8y^3 \div 2y$ **41.** $10y^2 \div 4y$ **42.** $8a \times 4a^3$
43. $(2x)^2 \times (3x)^3$ **44.** $4z^4 \times z^{-7}$ **45.** $6x^{-2} \div 3x^2$ **46.** $5y^3 \div 2y^{-2}$
47. $(x^2)^{\frac{1}{2}} \div (x^{\frac{1}{3}})^3$ **48.** $7w^{-2} \times 3w^{-1}$ **49.** $(2n)^4 \div 8n^0$ **50.** $4x^{\frac{3}{2}} \div 2x^{\frac{1}{2}}$

Example

Evaluate:

(a) $9^{\frac{1}{2}}$ (b) 5^{-1}
(c) $4^{-\frac{1}{2}}$ (d) $25^{\frac{3}{2}}$
(e) $(5^{\frac{1}{2}})^3 \times 5^{\frac{1}{2}}$ (f) 7^0

(a) $9^{\frac{1}{2}} = \sqrt{9} = 3$
(b) $5^{-1} = \frac{1}{5}$
(c) $4^{-\frac{1}{2}} = \dfrac{1}{4^{\frac{1}{2}}} = \dfrac{1}{\sqrt{4}} = \dfrac{1}{2}$
(d) $25^{\frac{3}{2}} = (\sqrt{25})^3 = 5^3 = 125$
(e) $(5^{\frac{1}{2}})^3 \times 5^{\frac{1}{2}} = 5^{\frac{3}{2}} \times 5^{\frac{1}{2}} = 5^2$
$= 25$
(f) $7^0 = 1 \left[\text{consider} \dfrac{7^3}{7^3} = 7^{3-3} = 7^0 = 1\right]$

Remember: $a^0 = 1$ for any non-zero value of a.

Exercise 13

Evaluate the following:

1. $3^2 \times 3$ **2.** 100^0 **3.** 3^{-2} **4.** $(5^{-1})^{-2}$
5. $4^{\frac{1}{2}}$ **6.** $16^{\frac{1}{2}}$ **7.** $81^{\frac{1}{2}}$ **8.** $8^{\frac{1}{3}}$
9. $9^{\frac{3}{2}}$ **10.** $27^{\frac{1}{3}}$ **11.** $9^{-\frac{1}{2}}$ **12.** $8^{-\frac{1}{3}}$
13. $1^{\frac{5}{2}}$ **14.** $25^{-\frac{1}{2}}$ **15.** $1000^{\frac{1}{3}}$ **16.** $2^{-2} \times 2^5$
17. $2^4 \div 2^{-1}$ **18.** $8^{\frac{2}{3}}$ **19.** $27^{-\frac{2}{3}}$ **20.** $4^{-\frac{3}{2}}$

21. $36^{\frac{1}{2}} \times 27^{\frac{1}{3}}$ **22.** $10\,000^{\frac{1}{4}}$ **23.** $100^{\frac{3}{2}}$ **24.** $(100^{\frac{1}{2}})^{-3}$

25. $(9^{\frac{1}{2}})^{-2}$ **26.** $(-16{\cdot}371)^0$ **27.** $81^{\frac{1}{4}} \div 16^{\frac{1}{4}}$ **28.** $(5^{-4})^{\frac{1}{2}}$

29. $1000^{-\frac{1}{3}}$ **30.** $(4^{-\frac{1}{2}})^2$ **31.** $8^{-\frac{2}{3}}$ **32.** $100^{\frac{5}{2}}$

33. $1^{\frac{4}{3}}$ **34.** 2^{-5} **35.** $(0{\cdot}01)^{\frac{1}{2}}$ **36.** $(0{\cdot}04)^{\frac{1}{2}}$

37. $(2{\cdot}25)^{\frac{1}{2}}$ **38.** $(7{\cdot}63)^0$ **39.** $3^5 \times 3^{-3}$ **40.** $(3\frac{3}{8})^{\frac{1}{3}}$

41. $(11\frac{1}{9})^{-\frac{1}{2}}$ **42.** $(\frac{1}{8})^{-2}$ **43.** $(\frac{1}{1000})^{\frac{2}{3}}$ **44.** $(\frac{9}{25})^{-\frac{1}{2}}$

45. $(10^{-6})^{\frac{1}{3}}$ **46.** $7^2 \div (7^{\frac{1}{2}})^4$ **47.** $(0{\cdot}0001)^{-\frac{1}{2}}$ **48.** $\dfrac{9^{\frac{1}{2}}}{4^{-\frac{1}{2}}}$

49. $\dfrac{25^{\frac{3}{2}} \times 4^{\frac{1}{2}}}{9^{-\frac{1}{2}}}$ **50.** $(-\frac{1}{7})^2 \div (-\frac{1}{7})^3$

Example

Simplify:

(a) $(2a)^3 \div (9a^2)^{\frac{1}{2}}$ (b) $(3ac^2)^3 \times 2a^{-2}$ (c) $(2x)^2 \div 2x^2$

(a) $(2a)^3 \div (9a^2)^{\frac{1}{2}} = 8a^3 \div 3a$
$= \frac{8}{3}a^2$

(b) $(3ac^2)^3 \times 2a^{-2} = 27a^3c^6 \times 2a^{-2}$
$= 54ac^6$

(c) $(2x)^2 \div 2x^2 = 4x^2 \div 2x^2$
$= 2$

Exercise 14

Rewrite without brackets:

1. $(5x^2)^2$ **2.** $(7y^3)^2$ **3.** $(10ab)^2$ **4.** $(2xy^2)^2$

5. $(4x^2)^{\frac{1}{2}}$ **6.** $(9y)^{-1}$ **7.** $(x^{-2})^{-1}$ **8.** $(2x^{-2})^{-1}$

9. $(5x^2y)^0$ **10.** $(\frac{1}{2}x)^{-1}$ **11.** $(3x)^2 \times (2x)^2$ **12.** $(5y)^2 \div y$

13. $(2x^{\frac{1}{2}})^4$ **14.** $(3y^{\frac{1}{3}})^3$ **15.** $(5x^0)^2$ **16.** $[(5x)^0]^2$

17. $(7y^0)^2$ **18.** $[(7y)^0]^2$ **19.** $(2x^2y)^3$ **20.** $(10xy^3)^2$

Simplify the following:

21. $(3x^{-1})^2 \div 6x^{-3}$ **22.** $(4x)^{\frac{1}{2}} \div x^{\frac{3}{2}}$ **23.** $x^2y^2 \times xy^3$ **24.** $4xy \times 3x^2y$

25. $10x^{-1}y^3 \times xy$ **26.** $(3x)^2 \times (\frac{1}{9}x^2)^{\frac{1}{2}}$ **27.** $z^3yx \times x^2yz$ **28.** $(2x)^{-2} \times 4x^3$

29. $(3y)^{-1} \div (9y^2)^{-1}$ **30.** $(xy)^0 \times (9x)^{\frac{3}{2}}$ **31.** $(x^2y)(2xy)(5y^3)$ **32.** $(4x^{\frac{1}{2}}) \times (8x^{\frac{3}{2}})$

33. $5x^{-3} \div 2x^{-5}$ **34.** $[(3x^{-1})^{-2}]^{-1}$ **35.** $(2a)^{-2} \times 8a^4$ **36.** $(abc^2)^3$

37. Write in the form 2^p (e.g. $4 = 2^2$):
(a) 32 (b) 128 (c) 64 (d) 1

38. Write in the form 3^q:
(a) $\frac{1}{27}$ (b) $\frac{1}{81}$ (c) $\frac{1}{3}$ (d) $9 \times \frac{1}{81}$

Evaluate, with $x = 16$ and $y = 8$.

39. $2x^{\frac{1}{2}} \times y^{\frac{1}{3}}$ **40.** $x^{\frac{1}{4}} \times y^{-1}$ **41.** $(y^2)^{\frac{1}{6}} \div (9x)^{\frac{1}{2}}$ **42.** $(x^2y^3)^0$

43. $x + y^{-1}$ **44.** $x^{-\frac{1}{2}} + y^{-1}$ **45.** $y^{\frac{1}{3}} \div x^{\frac{3}{4}}$ **46.** $(1000y)^{\frac{1}{3}} \times x^{-\frac{5}{2}}$

47. $(x^{\frac{1}{4}} + y^{-1}) \div x^{\frac{1}{4}}$ **48.** $x^{\frac{1}{2}} - y^{\frac{2}{3}}$ **49.** $(x^{\frac{3}{4}}y)^{-\frac{1}{3}}$ **50.** $\left(\dfrac{x}{y}\right)^{-2}$

Solve the equations for x.

51. $2^x = 8$
52. $3^x = 81$
53. $5^x = \frac{1}{5}$
54. $10^x = \frac{1}{100}$
55. $3^{-x} = \frac{1}{27}$
56. $4^x = 64$
57. $6^{-x} = \frac{1}{6}$
58. $100\,000^x = 10$
59. $12^x = 1$
60. $10^x = 0{\cdot}0001$
61. $2^x + 3^x = 13$
62. $(\frac{1}{2})^x = 32$
63. $5^{2x} = 25$
64. $1\,000\,000^{3x} = 10$
65. These two are more difficult. Use a calculator to find solutions correct to three significant figures.
(a) $x^x = 100$ (b) $x^x = 10\,000$

5.5 Inequalities

$x < 4$ means 'x is *less than 4*'
$y > 7$ means 'y is *greater than 7*'
$z \leqslant 10$ means 'z is *less than or equal to* 10'
$t \geqslant -3$ means 't is *greater than or equal to* -3'

Solving inequalities

We follow the same procedure used for solving equations except that when we multiply or divide by a *negative* number the inequality is *reversed.*

e.g. $4 > -2$
but multiplying by -2,
$-8 < 4$

Example

Solve the inequalities:

(a)
$$\begin{aligned} 2x - 1 &> 5 \\ 2x &> 5 + 1 \\ x &> \frac{6}{2} \\ x &> 3 \end{aligned}$$

(b)
$$\begin{aligned} 5 - 3x &\leqslant 1 \\ 5 &\leqslant 1 + 3x \\ 5 - 1 &\leqslant 3x \\ \frac{4}{3} &\leqslant x \end{aligned}$$

Exercise 15

Introduce one of the symbols $<$, $>$ or $=$ between each pair of numbers.

1. $-2, 1$
2. $(-2)^2, 1$
3. $\frac{1}{4}, \frac{1}{5}$
4. $0{\cdot}2, \frac{1}{5}$
5. $10^2, 2^{10}$
6. $\frac{1}{4}, 0{\cdot}4$
7. $40\%, 0{\cdot}4$
8. $(-1)^2, (-\frac{1}{2})^2$
9. $5^2, 2^5$
10. $3\frac{1}{3}, \sqrt{10}$
11. $\pi^2, 10$
12. $-\frac{1}{3}, -\frac{1}{2}$
13. $2^{-1}, 3^{-1}$
14. $50\%, \frac{1}{5}$
15. $1\%, 100^{-1}$

State whether the following are true or false:

16. $0{\cdot}7^2 > \frac{1}{2}$ **17.** $10^3 = 30$ **18.** $\frac{1}{8} > 12\%$

19. $(0{\cdot}1)^3 = 0{\cdot}0001$ **20.** $(-\frac{1}{5})^0 = -1$ **21.** $\frac{1}{5^2} > \frac{1}{2^5}$

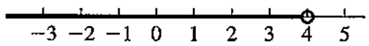
22. $(0{\cdot}2)^3 < (0{\cdot}3)^2$ **23.** $\frac{6}{7} > \frac{7}{8}$ **24.** $0{\cdot}1^2 > 0{\cdot}1$

Solve the following inequalities:

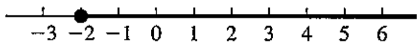
25. $x - 3 > 10$ **26.** $x + 1 < 0$ **27.** $5 > x - 7$ **28.** $2x + 1 \leqslant 6$
29. $3x - 4 > 5$ **30.** $10 \leqslant 2x - 6$ **31.** $5x < x + 1$ **32.** $2x \geqslant x - 3$
33. $4 + x < -4$ **34.** $3x + 1 < 2x + 5$ **35.** $2(x + 1) > x - 7$ **36.** $7 < 15 - x$
37. $9 > 12 - x$ **38.** $4 - 2x \leqslant 2$ **39.** $3(x - 1) < 2(1 - x)$ **40.** $7 - 3x < 0$

The number line

The inequality $x < 4$ is represented on the number line as

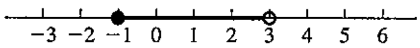


$x \geqslant -2$ is shown as



In the first case, 4 is *not* included so we have ○.
In the second case, -2 *is* included so we have ●.

$-1 \leqslant x < 3$ is shown as



Exercise 16

For questions **1** to **25**, solve each inequality and show the result on a number line.

1. $2x + 1 > 11$ **2.** $3x - 4 \leqslant 5$ **3.** $2 < x - 4$
4. $6 \geqslant 10 - x$ **5.** $8 < 9 - x$ **6.** $8x - 1 < 5x - 10$
7. $2x > 0$ **8.** $1 < 3x - 11$ **9.** $4 - x > 6 - 2x$
10. $\frac{x}{3} < -1$ **11.** $1 < x < 4$ **12.** $-2 \leqslant x \leqslant 5$
13. $1 \leqslant x < 6$ **14.** $0 \leqslant 2x < 10$ **15.** $-3 \leqslant 3x \leqslant 21$
16. $1 < 5x < 10$ **17.** $\frac{x}{4} > 20$ **18.** $3x - 1 > x + 19$
19. $7(x + 2) < 3x + 4$ **20.** $1 < 2x + 1 < 9$ **21.** $10 \leqslant 2x \leqslant x + 9$
22. $x < 3x + 2 < 2x + 6$ **23.** $10 \leqslant 2x - 1 \leqslant x + 5$ **24.** $x < 3x - 1 < 2x + 7$
25. $x - 10 < 2(x - 1) < x$

(Hint: in questions **20** to **25**, solve the two inequalities separately.)

For questions **26** to **35**, find the solutions, subject to the given condition.

26. $3a + 1 < 20$; a is a positive integer
27. $b - 1 \geqslant 6$; b is a prime number less than 20
28. $2e - 3 < 21$; e is a positive even number

29. $1 < z < 50$; z is a square number
30. $0 < 3x < 40$; x is divisible by 5
31. $2x > -10$; x is a negative integer
32. $x + 1 < 2x < x + 13$; x is an integer
33. $x^2 < 100$; x is a positive square number
34. $0 \leqslant 2z - 3 \leqslant z + 8$; z is a prime number
35. $\frac{a}{2} + 10 > a$; a is a positive even number
36. State the smallest integer n for which $4n > 19$.
37. Find an integer value of x such that $2x - 7 < 8 < 3x - 11$.
38. Find an integer value of y such that $3y - 4 < 12 < 4y - 5$.
39. Find any value of z such that $9 < z + 5 < 10$.
40. Find any value of p such that $9 < 2p + 1 < 11$.
41. Find a simple fraction q such that $\frac{4}{9} < q < \frac{5}{9}$.
42. Find an integer value of a such that $a - 3 \leqslant 11 \leqslant 2a + 10$.
43. State the largest prime number z for which $3z < 66$.
44. Find a simple fraction r such that $\frac{1}{3} < r < \frac{2}{3}$.
45. Find the largest prime number p such that $p^2 < 400$.
46. Illustrate on a number line the solution set of each pair of simultaneous inequalities:
    (a) $x < 6$; $-3 \leqslant x \leqslant 8$ (b) $x > -2$; $-4 < x < 2$
    (c) $2x + 1 \leqslant 5$; $-12 \leqslant 3x - 3$ (d) $3x - 2 < 19$; $2x \geqslant -6$
47. Find the integer n such that $n < \sqrt{300} < n + 1$.

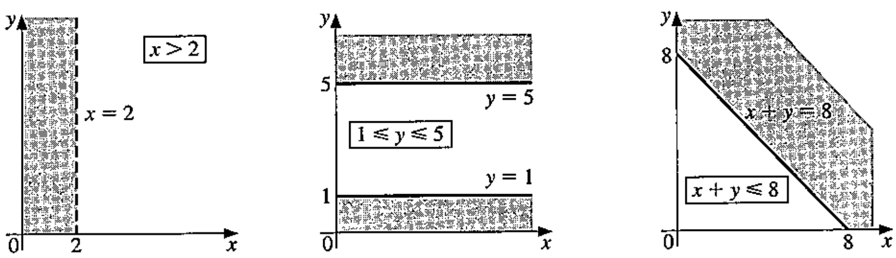
Graphical display

It is useful to represent inequalities on a graph, particularly where two variables are involved.

Drawing accurate graphs is explained in Unit 7.

Example

Draw a sketch graph and leave unshaded the area which represents the set of points that satisfy each of these inequalities:
(a) $x > 2$ (b) $1 \leqslant y \leqslant 5$ (c) $x + y \leqslant 8$



In each graph, the unwanted region is shaded so that the region representing the set of points is left clearly visible.

In (a), the line $x = 2$ is shown as a broken line to indicate that the points on the line are *not* included.

In (b) and (c), the lines $y = 1$, $y = 5$ and $x + y = 8$ are shown as solid lines because points on the line *are* included in the solution set.

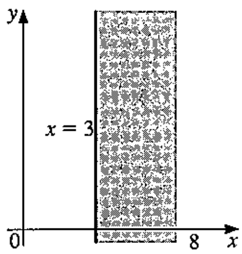
An inequality can thus be regarded as a set of points, for example, the unshaded region in (c) may be described as

$$\{(x, y): x + y \leqslant 8\}$$

i.e. the set of points (x, y) such that $x + y \leqslant 8$.

Exercise 17

In questions **1** to **9** describe the region left unshaded.

1.


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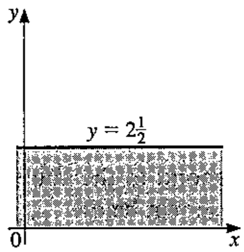

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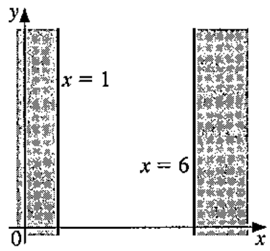

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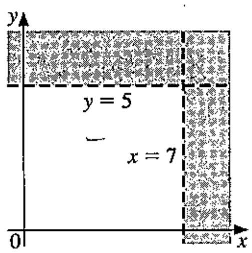

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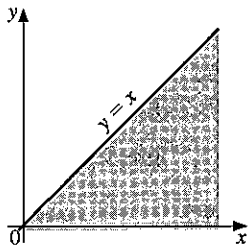

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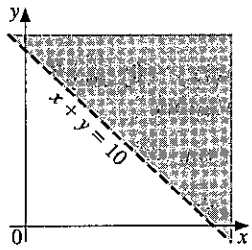

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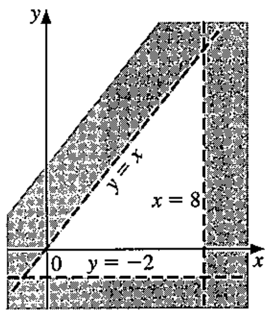
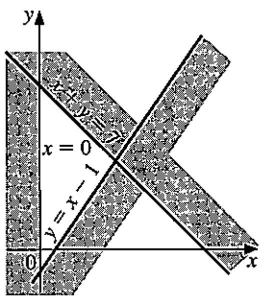
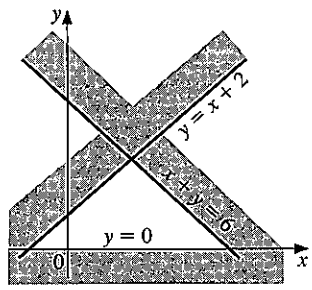


8.


9.


For questions **10** to **27**, draw a sketch graph similar to those above and indicate the set of points which satisfy the inequalities by shading the unwanted regions.

10. $2 \leqslant x \leqslant 7$

11. $0 \leqslant y \leqslant 3\frac{1}{2}$

12. $-2 < x < 2$

13. $x < 6$ and $y \leqslant 4$

14. $0 < x < 5$ and $y < 3$

15. $1 \leqslant x \leqslant 6$ and $2 \leqslant y \leqslant 8$

16. $-3 < x < 0$ and $-4 < y < 2$

17. $y \leqslant x$

18. $x + y < 5$

19. $y > x + 2$ and $y < 7$

20. $x \geqslant 0$ and $y \geqslant 0$ and $x + y \leqslant 7$

21. $x \geqslant 0$ and $x + y < 10$ and $y > x$

22. $8 \geqslant y \geqslant 0$ and $x + y > 3$

23. $x + 2y < 10$ and $x \geqslant 0$ and $y \geqslant 0$

24. $3x + 2y \leqslant 18$ and $x \geqslant 0$ and $y \geqslant 0$

25. $x \geqslant 0, y \geqslant x - 2, x + y \leqslant 10$

26. $3x + 5y \leqslant 30$ and $y > \frac{x}{2}$

27. $y \geqslant \frac{x}{2}, y \leqslant 2x$ and $x + y \leqslant 8$

5.6 Linear programming

In most linear programming problems, there are two stages:
1. to interpret the information given as a series of simultaneous inequalities and display them graphically.
2. to investigate some characteristic of the points in the unshaded solution set.

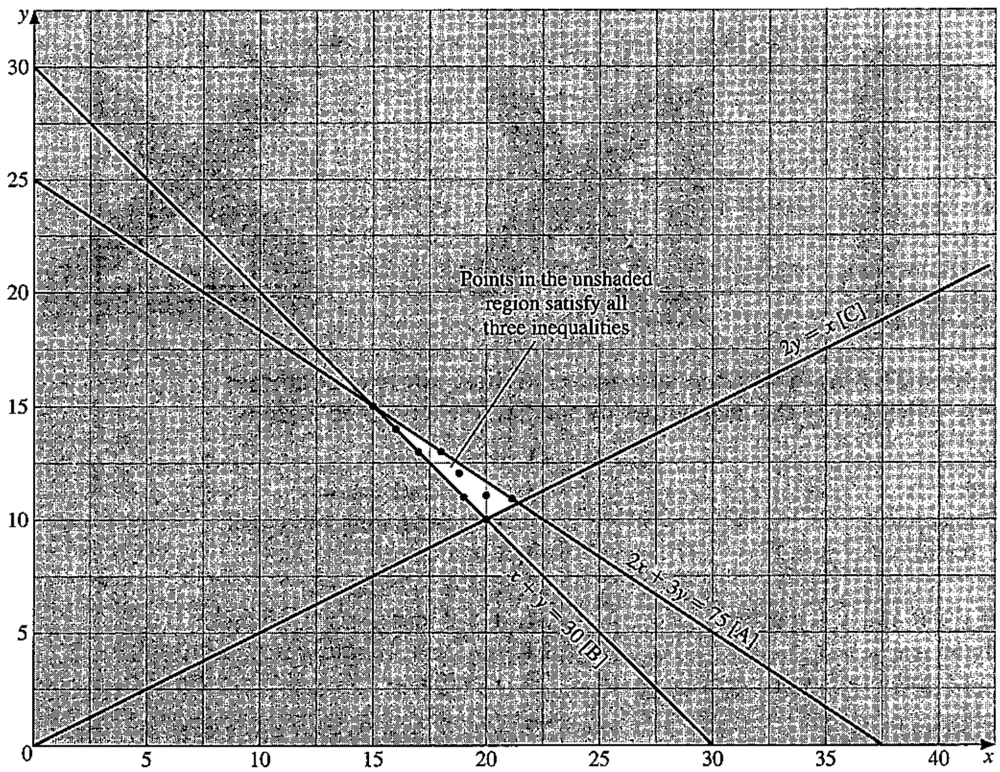
Example

A shopkeeper buys two types of dog food for his shop: Bruno at 40c a tin and Blaze at 60c a tin. He has \$15 available and decides to buy at least 30 tins altogether. He also decides that at least one third of the tins should be Blaze. He buys x tins of Bruno and y tins of Blaze.

(a) Write down three inequalities which correspond to the above conditions.
(b) Illustrate these inequalities on a graph, as shown below.
(c) He makes a profit of 10c a tin on Bruno and a profit of 20c a tin on Blaze. Assuming he can sell all his stock, find how many tins of each type he should buy to maximise his profit and find that profit.

(a) Cost $40x + 60y \leqslant 1500$ $\quad 2x + 3y \leqslant 75$ ... [line A on graph]
Total number $\quad x + y \geqslant 30$ ... [line B on graph]
At least one third Blaze $\dfrac{y}{x} \geqslant \dfrac{1}{2}$ $\quad 2y \geqslant x$ ... [line C on graph]

(b) The graph below shows these three equations.



(c) The table below shows the points on the graph in the unshaded region together with the corresponding figure for the profit.

The points marked * will clearly not provide a maximum profit.

x	15	16	17*	18	19	19*	20*	20*	21
y	15	14	13	13	12	11	11	10	11
profit	150	160		180	190				210
	+300	+280		+260	+240				+220
	450c	440c		440c	430c				430c

Conclusion: he should buy 15 tins of Bruno and 15 tins of Blaze. His maximised profit is then 450c.

Exercise 18

For questions **1** to **3**, draw an accurate graph to represent the inequalities listed, using shading to show the unwanted regions.

1. $x + y \leqslant 11$; $y \geqslant 3$; $y \leqslant x$.
Find the point having whole number coordinates and satisfying these inequalities which gives:
(a) the maximum value of $x + 4y$
(b) the minimum value of $3x + y$

2. $3x + 2y > 24$; $x + y < 12$; $y < \frac{1}{2}x$; $y > 1$.
Find the point having whole number coordinates and satisfying these inequalities which gives:
(a) the maximum value of $2x + 3y$
(b) the minimum value of $x + y$

3. $3x + 2y \leqslant 60$; $x + 2y \leqslant 30$; $x \geqslant 10$; $y \geqslant 0$.
Find the point having whole number coordinates and satisfying these inequalities which gives:
(a) the maximum value of $2x + y$
(b) the maximum value of xy

4. Kojo is given \$1·20 to buy some peaches and apples. Peaches cost 20c each, apples 10c each. He is told to buy at least 6 individual fruits, but he must not buy more apples than peaches.
Let x be the number of peaches Kojo buys.
Let y be the number of apples Kojo buys.
(a) Write down three inequalities which must be satisfied.
(b) Draw a linear programming graph and use it to list the combinations of fruit that are open to Kojo.

5. Laura is told to buy some melons and oranges. Melons are 50c each and oranges 25c each, and she has \$2 to spend. She must not buy more than 2 melons and she must buy at least 4 oranges. She is also told to buy at least 6 fruits altogether.
Let x be the number of melons.
Let y be the number of oranges.

(a) Write down four inequalities which must be satisfied.
(b) Draw a graph and use it to list the combinations of fruit that are open to Laura.

6. A chef is going to make some fruit cakes and sponge cakes. He has plenty of all ingredients except for flour and sugar. He has only 2000 g of flour and 1200 g of sugar.
A fruit cake uses 500 g of flour and 100 g of sugar.
A sponge cake uses 200 g of flour and 200 g of sugar.
He wishes to make *more than* 4 cakes altogether.
Let the number of fruit cakes be x.
Let the number of sponge cakes be y.
(a) Write down three inequalities which must be satisfied.
(b) Draw a graph and use it to list the possible combinations of fruit cakes and sponge cakes which he can make.

7. Kwame has a spare time job spraying cars and vans. Vans take 2 hours each and cars take 1 hour each. He has 14 hours available per week. He has an agreement with one firm to do 2 of their vans every week. Apart from that he has no fixed work.
Kwame's permission to use his back garden contains the clause that he must do at least twice as many cars as vans.
Let x be the number of vans sprayed each week.
Let y be the number of cars sprayed each week.
(a) Write down three inequalities which must be satisfied.
(b) Draw a graph and use it to list the possible combinations of vehicles which Kwame can spray each week.

8. The manager of a football team has \$100 to spend on buying new players. He can buy defenders at \$6 each or forwards at \$8 each. There must be at least 6 of each sort. To cover for injuries he must buy at least 13 players altogether. Let x represent the number of defenders he buys and y the number of forwards.
(a) In what ways can he buy players?
(b) If the wages are \$10 per week for each defender and \$20 per week for each forward, what is the combination of players which has the lowest wage bill?

9. A tennis-playing golfer has \$15 to spend on golf balls (x) costing \$1 each and tennis balls (y) costing 60c each. He must buy at least 16 altogether and he must buy *more* golf balls than tennis balls.
(a) What is the greatest number of balls he can buy?
(b) After using them, he can sell golf balls for 10c each and tennis balls for 20c each. What is his maximum possible income from sales?

10. A travel agent has to fly 1000 people and 35 000 kg of baggage from Hong Kong to Shanghai. Two types of aircraft are available:
A which takes 100 people and 2000 kg of baggage, or B which takes 60 people and 3000 kg of baggage. He can use no more than 16 aircraft altogether. Write down three inequalities which must be satisfied if he uses x of A and y of B.

(a) What is the smallest number of aircraft he could use?
(b) If the hire charge for each aircraft A is \$10 000 and for each aircraft B is \$12 000, find the cheapest option available to him.
(c) If the hire charges are altered so that each A costs \$10 000 and each B costs \$20 000, find the cheapest option now available to him.

11. A farmer has to transport 20 people and 32 sheep to a market. He can use either Fiats (x) which take 2 people and 1 sheep, or Rolls Royces (y) which take 2 people and 4 sheep.
He must not use more than 15 cars altogether.
(a) What is the lowest total numbers of cars he could use?
(b) If it costs \$10 to hire each Fiat and \$30 for each Rolls Royce, what is the *cheapest* solution?

12. A shop owner wishes to buy up to 20 televisions for stock. He can buy either type A for \$150 each or type B for \$300 each. He has a total of \$4500 he can spend. He must have at least 6 of each type in stock. If he buys x of type A and y of type B, write down 4 inequalities which must be satisfied and represent the information on a graph.
(a) If he makes a profit of \$40 on each of type A and \$100 on each of type B, how many of each should he buy for maximum profit?
(b) If the profit is \$80 on each of type A and \$100 on each of type B, how many of each should he buy now?

13. A farmer needs to buy up to 25 cows for a new herd. He can buy either brown cows (x) at \$50 each or black cows (y) at \$80 each and he can spend a total of no more than \$1600. He must have at least 9 of each type.
On selling the cows he makes a profit of \$50 on each brown cow and \$60 on each black cow. How many of each sort should he buy for maximum profit?

14. The manager of a car park allows $10\,m^2$ of parking space for each car and $30\,m^2$ for each lorry. The total space available is $300\,m^2$. He decides that the maximum number of vehicles at any time must not exceed 20 and he also insists that there must be at least as many cars as lorries. If the number of cars is x and the number of lorries is y, write down three inequalities which must be satisfied.
(a) If the parking charge is \$1 for a car and \$5 for a lorry, find how many vehicles of each kind he should admit to maximise his income.
(b) If the charges are changed to \$2 for a car and \$3 for a lorry, find how many of each kind he would be advised to admit.

Revision exercise 5A

1. Express the following as single fractions:

(a) $\dfrac{x}{4}+\dfrac{x}{5}$ (b) $\dfrac{1}{2x}+\dfrac{2}{3x}$

(c) $\dfrac{x+2}{2}+\dfrac{x-4}{3}$ (d) $\dfrac{7}{x-1}-\dfrac{2}{x+3}$

2. (a) Factorise x^2-4

(b) Simplify $\dfrac{3x-6}{x^2-4}$

3. Given that $s-3t=rt$, express:

(a) s in terms of r and t (b) r in terms of s and t

(c) t in terms of s and r.

4. (a) Given that $x-z=5y$, express z in terms of x and y.

(b) Given that $mk+3m=11$, express m in terms of k.

(c) For the formula $T=C\sqrt{z}$, express z in terms of T and C.

5. It is given that $y=\dfrac{k}{x}$ and that $1\leqslant x\leqslant 10$.

(a) If the smallest possible value of y is 5, find the value of the constant k.

(b) Find the largest possible value of y.

6. Given that y varies as x^2 and that $y=36$ when $x=3$, find:

(a) the value of y when $x=2$ (b) the value of x when $y=64$.

7. (a) Evaluate: (i) $9^{\frac{1}{2}}$ (ii) $8^{\frac{2}{3}}$ (iii) $16^{-\frac{1}{2}}$

(b) Find x, given that

(i) $3^x=81$ (ii) $7^x=1$.

8. List the integer values of x which satisfy.

(a) $2x-1<20<3x-5$ (b) $5<3x+1<17$.

9. Given that $t=k\sqrt{(x+5)}$, express x in terms of t and k.

10. Given that $z=\dfrac{3y+2}{y-1}$, express y in terms of z.

11. Given that $y=\dfrac{k}{k+w}$

(a) Find the value of y when $k=\frac{1}{2}$ and $w=\frac{1}{3}$

(b) Express w in terms of y and k.

12. On a suitable sketch graph, identify clearly the region A defined by $x\geqslant 0$, $x+y\leqslant 8$ and $y\geqslant x$.

13. Without using a calculator, calculate the value of:

(a) $9^{-\frac{1}{2}}+(\frac{1}{8})^{\frac{1}{3}}+(-3)^0$ (b) $(1000)^{-\frac{1}{3}}-(0{\cdot}1)^2$

14. It is given that $10^x=3$ and $10^y=7$. What is the value of 10^{x+y}?

15. Make x the subject of the following formulae:

(a) $x + a = \dfrac{2x - 5}{a}$ (b) $cz + ax + b = 0$ (c) $a = \sqrt{\left(\dfrac{x+1}{x-1}\right)}$

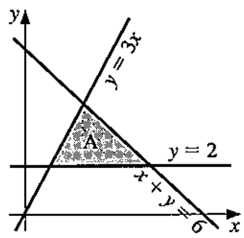
16. Write the following as single fractions:

(a) $\dfrac{3}{x} + \dfrac{1}{2x}$ (b) $\dfrac{3}{a-2} + \dfrac{1}{a^2 - 4}$ (c) $\dfrac{3}{x(x+1)} - \dfrac{2}{x(x-2)}$

17. p varies jointly as the square of t and inversely as s. Given that $p = 5$ when $t = 1$ and $s = 2$, find a formula for p in terms of t and s.

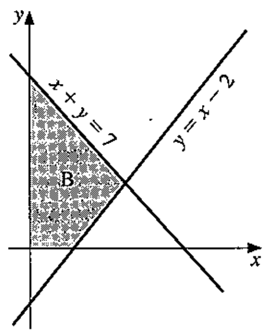
18. A positive integer r is such that $pr^2 = 168$, where p lies between 3 and 5. List the possible values of r.

19. The shaded region A is formed by the lines $y = 2$, $y = 3x$ and $x + y = 6$. Write down the three inequalities which define A.



20. The shaded region B is formed by the lines $x = 0$, $y = x - 2$ and $x + y = 7$.

Write down the four inequalities which define B.



21. In the diagram, the solution set $-1 \leqslant x < 2$ is shown on a number line.

Illustrate, on similar diagrams, the solution set of the following pairs of simultaneous inequalities.

(a) $x > 2$, $x \leqslant 7$ (b) $4 + x \geqslant 2$, $x + 4 < 10$

(c) $2x + 1 \geqslant 3$, $x - 3 \leqslant 3$.

22. In a laboratory we start with 2 cells in a dish. The number of cells in the dish doubles every 30 minutes.

(a) How many cells are in the dish after four hours?

(b) After what time are there 2^{13} cells in the dish?

(c) After $10\frac{1}{2}$ hours there are 2^{22} cells in the dish and an experimental fluid is added which eliminates half of the cells. How many cells are left?

Examination exercise 5B

1. (a) Find the value of $81^{\frac{3}{4}}$.

(b) Simplify $\dfrac{3x^{-\frac{2}{3}}}{6x^{\frac{1}{3}}}$. N 95 2

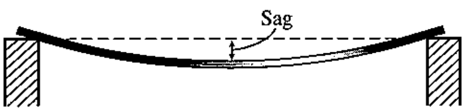
2. When $x = 27$, $y = \dfrac{1}{3}$ and $z = 2$, find the value of:

(a) $8x^{\frac{1}{3}}$ (b) $\left(\dfrac{y}{z}\right)^{-2}$ (c) $(xy)^0$ J 96 2

3. A particle moving in a circle has an acceleration a which is inversely proportional to the radius r of the circle.
When $r = 24$, $a = 2$.
(a) Find an equation connecting a and r.
(b) Calculate r when $a = 10$. J 95 2

4. Find the values of p, q and r.
(a) $\sqrt{x^{36}} = x^p$ (b) $10^q = 1$ (c) $r^{-\frac{1}{2}} = \frac{1}{4}$ J 98 2

5.

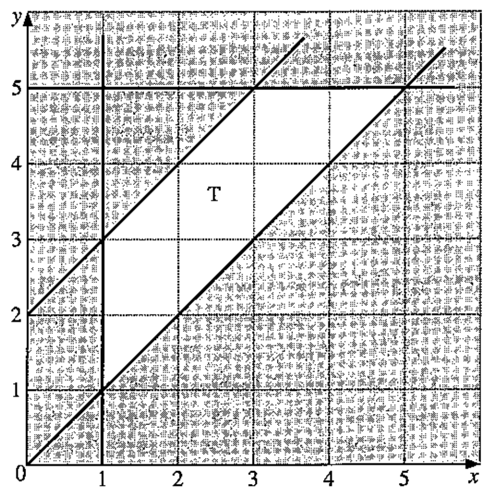


An engineer tests iron rods.
He measures the sag for different lengths of rod.
The results are as follows.

Length of rod (x metres)	0	1	2	3	4
Sag (y millimetres)	0	0·5	4	13·5	32

He knows that $y \propto x^n$, where n is a positive integer. Find the value of n. N 96 2

6. The trapezium T is defined by four inequalities.
One is $y \geqslant x$. Write down the other three inequalities. J 98 2



7. Arnie and Bernie are tailors. They make x jackets and y suits each week. Arnie does all the cutting, and Bernie does all the sewing. To make a jacket takes 5 hours of cutting and 4 hours of sewing. To make a suit takes 6 hours of cutting and 10 hours of sewing. Neither tailor works for more than 60 hours a week.

(a) For the sewing, show that:

$$2x + 5y \leqslant 30$$

(b) Write down another inequality in x and y for the cutting.

(c) They make at least 8 jackets each week. Write down another inequality.

(d) (i) Draw axes from 0 to 16, using 1 cm to represent 1 unit on each axis.

(ii) On your grid, show the information in parts (a), (b) and (c). Shade the unwanted regions.

(e) The profit on a jacket is \$30 and on a suit is \$100. Calculate the maximum profit that Arnie and Bernie can make in a week.

N 96 4

8. A ferry has a deck area of $3600\,\text{m}^2$ for parking cars and trucks. Each car takes up $20\,\text{m}^2$ of deck area and each truck takes up $80\,\text{m}^2$. On one trip, the ferry carries x cars and y trucks.

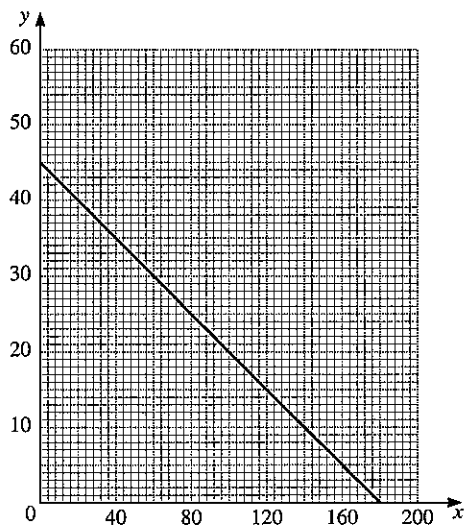
(a) Show that this information leads to the inequality $x + 4y \leqslant 180$.

(b) The charge for the trip is \$25 for a car and \$50 for a truck. The total amount of money taken is \$3000. Write down an equation to represent this information and simplify it.

(c) The line $x + 4y = 180$ is drawn on this grid.

(i) Draw, on a copy of the grid, the graph of your equation in part (b).

(ii) Write down a possible number of cars and a possible number of trucks on the trip, which together satisfy both conditions.



J 03 2

9. Solve the inequality

$$3 < 2x - 5 < 7$$

J 03 2

10. Work out as a single fraction

$$\frac{2}{x-3} - \frac{1}{x+4}$$

J 03 2